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Relationship Between Volterra Series and Generalized Power Series

MICHAEL STEER, PETER J. KHAN, AND RODNEY S. TUCKER

Abstract—The relationship between the Volterra nonlinear transfer functions of a system and the elements of its generalized power series is established. A formula is derived which enables the Volterra nonlinear transfer functions to be obtained from the power series expansion of the nonlinear system.

Introduction

Volterra series expansions can be used in the analysis of weakly nonlinear systems with single-valued, and possibly frequency-dependent, input–output characteristics. The analysis procedure requires the algebraic determination of Volterra nonlinear transfer functions [1]. The use of Volterra series is limited to weakly nonlinear systems because of the algebraic complexity of calculating high-order nonlinear transfer functions.

Nonlinear systems also can be analyzed using a power series expansion of the system characteristics. Most power-series-based analyses have been limited to memoryless systems. The range of systems that can be handled by power series expansions is greatly increased by using complex coefficients and frequency-dependent time delays. This gives rise to the generalized power series introduced by Steer and Khan [2].

Here we develop a formula which enables the Volterra nonlinear transfer functions of a system to be obtained from the power series expansion of the input–output characteristics of the system. This result is significant, as physical knowledge of the nonlinear elements comprising a system can yield the power series parameters of the system [3].

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M. B. Steer was with the University of Queensland, St. Lucia, Queensland, Australia. He is now with the Department of Electrical and Computer Engineering, North Carolina State University, Raliegh, NC 27650.

P. J. Khan and R. S. Tucker are with the Department of Electrical Engineering, University of Queensland, St. Lucia, Queensland, 4067, Australia.

Mathematical Development

The relationship developed here between Volterra series and generalized power series is obtained by comparing two formulas for the phasor of an output component of a system. One of these formulas is expressed in terms of Volterra nonlinear transfer functions and the other in terms of the parameters of a generalized power series.

In considering a system with a two-tone input, Bussgang *et al.* [1] derived a formula, involving Volterra nonlinear transfer functions, for the output components of a system. Using a more general derivation their result has been extended to a system excited by N sinusoidal inputs.

With the input $x(t)$ expressed as a sum of sinusoids

$$x(t) = \sum_{k=1}^{N} x_k(t) \tag{1}$$

where

$$x_k(t) = |x_k| \cos(\omega_k t + \phi_k)$$

the phasor $B[n_1,\cdots,n_N; S_1,\cdots,S_N]$ of a component of the output of order $n + 2\sigma$ is given by

$$B[n_1,\cdots,n_N; S_1,\cdots,S_N] = \sum_{\substack{S_1, S_2,\cdots,S_N \\ (S_1+S_2+\cdots+S_N)=\sigma}} \Bigg\{ \frac{(n+2\sigma)!}{2^{n+2\sigma-1}} \prod_{k=1}^{N} \frac{1}{(|n_k|+S_k)!S_k!} \cdot \left[\prod_{k=1}^{N} (X_k')^{|n_k|} |X_k|^{2S_k} \right] H_{n+2\sigma}[n_1,\cdots,n_N; S_1,\cdots,S_N] \Bigg\} \tag{2}$$

where $H_{n+2\sigma}[n_1,\cdots,n_N; S_1,\cdots,S_N]$ is the Volterra nonlinear transfer function of order $n + 2\sigma$; X_k is the phasor form of x_k; $X_k' = X_k$ if n_k is positive and $X_k' = X_k^*$ if n_k is negative; S_k is a summation variable; $\sigma = S_1 + \cdots + S_N$; and $n = n_1 + \cdots + n_N$. The n_k's are a set of integers which defines the frequency of the output phasor according to

$$\omega = \sum_{k=1}^{N} n_k \omega_k.$$

$B[n_1,\cdots,n_N; S_1,\cdots,S_N]$ is the phasor of an intermodulation product of the input components, the x_k's. Hence the set of n_k's is called an intermodulation product description (IPD).

In (2), a new shorthand notation is used for the arguments of the nonlinear transfer function and the output phasor. Bussgang *et al.* use the argument list

$$(\omega_1, \omega_1, \cdots, \omega_{-1}, \omega_{-1}, \cdots, \omega_k, \omega_k, \cdots, \omega_{-k}, \omega_{-k}, \cdots, \omega_{-N}, \omega_{-N})$$

where ω_k is repeated $|n_k| + S_k$ times if n_k is positive, or S_k times otherwise; ω_{-k} $(= -\omega_k)$ is repeated S_k times if n_k is positive or $|n_k| + S_k$ times otherwise. In the present work the shorthand notation for this list is

$$[n_1,\cdots,n_N; S_1,\cdots,S_N].$$

If an output component of radian frequency ω is defined by a set of n_k's then $-\omega$ is defined by the negative of this set (that is, replacing each n_k by $-n_k$). Thus the phasor of an output component of radian frequency $-\omega$ is

$$B[-n_1,\cdots,-n_N;S_1,\cdots,S_N] = \sum_{\substack{S_1,S_2,\cdots,S_N \\ (S_1+S_2+\cdots+S_N)=\sigma}} \left\{ \frac{(n+2\sigma)!}{2^{n+2\sigma-1}} \prod_{k=1}^{N} \frac{1}{(|n_k|+S_k)!S_k!} \cdot \left[\prod_{k=1}^{N} (X_k')^{*|n_k|}|X_k|^{2S_k}\right] H_{n+2\sigma}[-n_1,\cdots,-n_N;S_1,\cdots,S_N]\right\}. \quad (3)$$

The positive-frequency and negative-frequency nonlinear transfer functions (defined by an IPD and the negative of this IPD, respectively) are related by [1]

$$H_{n+2\sigma}[-n_1,\cdots,-n_N;S_1,\cdots,S_N] = H^*_{n+2\sigma}[n_1,\cdots,n_N;S_1,\cdots,S_N].$$

Thus, from (2) and (3), the positive and negative frequency components of the output are related by

$$B[-n_1,\cdots,-n_N;S_1,\cdots,S_N] = B^*[n_1,\cdots,n_N;S_1,\cdots,S_N].$$

This enables us to write down an expression for an intermodulation product. The output intermodulation product defined by one IPD is

$$|V_\omega|\cos(\omega t+\phi) = \tfrac{1}{2}V_\omega \exp(j\omega t) + \tfrac{1}{2}V_\omega^* \exp(-j\omega t) \quad (4)$$

where

$$V_\omega = \sum_{\sigma=0}^{\infty} B[n_1,\cdots,n_N;S_1,\cdots,S_N] \quad (5)$$

and

$$V_\omega^* = \sum_{\sigma=0}^{\infty} B[-n_1,\cdots,-n_N;S_1,\cdots,S_N].$$

The component of the total output of radian frequency ω is the summation of (4) over all IPD's for that frequency.

The generalized power series expansion of the output of a system is [2]

$$y(t) = \sum_{i=1}^{I} A_i \sum_{l=0}^{\infty} a_{l,i}\left[\sum_{k=1}^{N} b_{k,i}x_k(t-\tau_{k,l,i})\right]^l \quad (6)$$

where $y(t)$ is the output of the system; l is the order of the power series terms; $\tau_{k,l,i}$ is a time delay that depends on frequency and order; $a_{l,i}$ is a complex coefficient; and $b_{k,i}$ is a real coefficient. For this system the phasor of an output intermodulation product is (5) where [2]

$$V_\omega = \mathrm{Re}\left[\epsilon_n\left(\prod_{k=1}^{N} X_k^{\prime|n_k|}\right)\cdot T\right]_\omega \quad (7)$$

$$T = \sum_{\sigma=0}^{\infty} \frac{(n+2\sigma)!}{2^{(n+2\sigma)}}\cdot z \quad (8)$$

and

$$z = \sum_{\substack{S_1,\cdots,S_N \\ S_1+\cdots+S_N=\sigma}} \left(\prod_{k=1}^{N} \frac{|X_k'|^{2S_k}}{S_k!(|n_k|+S_k)!}\right) \cdot \sum_{i=1}^{I}\left(A_i a_{n+2\sigma,i} R_{n+2\sigma,i} \prod_{k=1}^{N} (b_{k,i})^{(|n_k|+2S_k)}\right). \quad (9)$$

In (7)–(9) $\epsilon_n(\epsilon_n = 1, n = 0; \epsilon_n = 2, n \neq 0)$ is the Neuman factor; the function $\mathrm{Re}[\cdot]_\omega$ is defined so that the value of the function is the real part of the expression in brackets for $\omega = 0$, but for $\omega \neq 0$ the function is equal to the expression in brackets; and $R_{n+2\sigma,i}$ is a function of time delays

$$R_{n+2\sigma,i} = \exp\left(-j\sum_{k=1}^{N} n_k\omega_k\tau_{k,(n+2\sigma),i}\right) = \prod_{k=1}^{N}\left(\mathrm{T}'_{k,(n+2\sigma),i}\right)^{|n_k|}.$$

We can now obtain the desired relationship between Volterra nonlinear transfer functions and the parameters of a generalized power series. By comparing (5) and (7) we obtain

$$H_{n+2\sigma}[n_1,\cdots,n_N;S_1,\cdots,S_N] = \mathrm{Re}\left[\epsilon_n \sum_{i=1}^{I} A_i a_{n+2\sigma,i} R_{n+2\sigma,i} \prod_{k=1}^{N} (b_{k,i})^{(|n_k|+2S_k)}\right]_\omega. \quad (10)$$

Discussion and Conclusions

The significance of the above result is that if a system can be described by a generalized power series, then the Volterra nonlinear transfer function of the system can readily be found. This is an important result because the algebraic determination of Volterra nonlinear transfer function of order higher than three is difficult. The analysis of a nonlinear system described by a generalized power series can be performed using the output formula based on the power series parameters (7). However, to analyze interconnected nonlinear subsystems it is usual to obtain the Volterra nonlinear transfer functions of the individual subsystems.

To illustrate the relationship between a Volterra nonlinear transfer function and the parameters of a power series, consider a system described by a power series with real coefficients and no time delays. The output of this system is given by

$$y = \sum_{l=1}^{\infty} a_l x^l$$

and, from (10), a Volterra nonlinear transfer functions of this system is related to a power series coefficient by

$$H_{n+2\sigma}[n_1,\cdots,n_N;S_1,\cdots,S_N] = \mathrm{Re}[\epsilon_n a_{n+2\sigma}].$$

In the derivation of the Volterra series based output formula, Bussgang *et al.* imposed the condition that there is no dc input to the system. This simplifies their derivation by eliminating the intermodulation products resulting from the mixing of the sinusoidal inputs with the dc input. However, the generalized power series formulation [2] includes dc as an input to the system. Thus the relationship established here between Volterra series and generalized power series includes dc input terms.

References

[1] S. L. Bussgang, L. Ehrman, and J. W. Graham, "Analysis of non-linear systems with multiple inputs," *Proc. IEEE*, vol. 62, pp. 1088–1119, Aug. 1974.
[2] M. B. Steer and P. J. Khan, "An algebraic formula for the output of a system with multifrequency excitation," *Proc. IEEE*, vol. 71, no. 1, pp. 177–179, Jan. 1983.
[3] R. S. Tucker, "Third-order intermodulation distortion and gain compression in GaAs FET's," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-27, pp. 400–408, May 1979.

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