An Algebraic Formula for the Output of a System with Large-Signal, Multifrequency Excitation

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Abstract—A formula is derived for the output components of a nonlinearity which can be described by a power series, with complex coefficients and frequency-dependent time delays, when the input is a sum of sinusoids.

Introduction

The output $y(t)$ for a system having a multifrequency input $x(t)$, where

$$x(t) = \sum_{k=1}^{N} x_k(t)$$

and

$$x_k(t) = |x_k| \cos(\omega_k t + \phi)$$

has been found by Sea and Vacroux [1], [2] for the case when

$$y(t) = A \sum_{l=0}^{\infty} a_l (x(t))^l \tag{1}$$

and by Price and Khan [3] for

$$y(t) = A(1 + x(t))^{\alpha} \tag{2}$$

where α is a noninteger.

Here we consider the general nonlinear system described by a complex power series with frequency-dependent time delays such that

$$y = \sum_{i=1}^{I} A_i \left(\sum_{l=0}^{\infty} a_{l,i} f(i, l, x) \right) \tag{3}$$

with

$$f(i, l, x) = \left(\sum_{k=1}^{N} b_{k,i} x_k(t - \tau_{k,l,i}) \right)^l$$

and both the A_i and $a_{l,i}$ coefficients are complex, thus indicating a phase shift, while the $b_{k,i}$ coefficients are real. This representation is applicable to a broad class of frequency-dependent nonlinearities, and has recently been used to analyze distortion in microwave FET's [4] and distortion due to phase nonlinearities [5].

Mathematical Development

A component of (3) can be written as

$$\begin{aligned} x_k(t - \tau_{k,l,i}) &= |x_k| \cos(\omega_k t + \phi_k - \omega_k \tau_{k,l,i}) \\ &= \tfrac{1}{2} X_k \mathrm{T}_{k,l,i} e^{j\omega_k t} + \tfrac{1}{2} X_k^* \mathrm{T}_{k,l,i}^* e^{-j\omega_k t} \end{aligned}$$

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where X_k is the phasor of x_k and

$$\Upsilon_{k,l,i} = \exp(-j\omega_k \tau_{k,l,i}).$$

$\Upsilon_{k,l,i}$ and X_k are defined separately as this results in a final expression that can be more efficiently computed.

Using the multinomial expansion theorem, we have

$$f(i,l,x) = \sum_{\substack{q_1,q_2,\cdots q_N, r_1, r_2, \cdots r_N \\ q_1+\cdots+q_N+r_1+\cdots r_N = l}} \left[\exp\left(j\sum_{k=1}^{N}(q_k - r_k)\omega_k t\right)\right] l! \prod_{k=1}^{N}\left(\frac{(\frac{1}{2}b_{k,i})^{q_k+r_k}(\frac{1}{2}X_k)^{q_k}(\frac{1}{2}X_k^*)^{r_k}(\Upsilon_{k,l,i})^{q_k}(\Upsilon_{k,l,i}^*)^{r_k}}{q_k! r_k!}\right). \tag{4}$$

In this equation the frequency of each component is given by

$$\omega = \sum_{k=1}^{N} n_k \omega_k. \tag{5}$$

Here the n_k's is a set of integers (an intermodulation product description (IPD)) which fixes the difference between q_k and r_k.

If we let $s_k (\geq 0)$ be the smaller of q_k and r_k, then $s_k + |n_k|$ is the larger. Defining

$$\sigma = \sum_{k=1}^{N} s_k \quad \text{and} \quad n = \sum_{k=1}^{N} |n_k|$$

then for $n \neq 0$, one IPD of (4) can be written

$$\begin{aligned}(\tfrac{1}{2}C)\exp(j\omega t) + (\tfrac{1}{2}C)^*\exp(-j\omega t) \\ = (\tfrac{1}{2}V'_\omega)\exp(j\omega t) + (\tfrac{1}{2}V'_\omega)^*\exp(-j\omega t), &\quad \omega \neq 0 \\ = V'_\omega, &\quad \omega = 0\end{aligned} \tag{6}$$

where

$$C = 2\sum_{\substack{s_1,s_2,\cdots s_N \\ 2(s_1+s_2+\cdots+s_N)+n=l}} l! \prod_{k=1}^{N} \cdot\left(\left|\frac{1}{2}b_{k,i}\right|^{2s_k+|n_k|} \frac{|\frac{1}{2}X_k|^{2s_k}(\frac{1}{2}X'_k)^{|n_k|}\cdot(\Upsilon'_{k,l,i})^{|n_k|}}{(s_k)!\cdot(s_k+|n_k|)!}\right) \tag{7}$$

$$X'_k = \begin{cases} X_k, & \text{if } n_k > 0 \\ X_k^*, & \text{if } n_k < 0\end{cases}$$

$$\Upsilon'_{k,l,i} = \begin{cases} \Upsilon_{k,l,i}, & \text{if } n_k > 0 \\ \Upsilon^*_{k,l,i}, & \text{if } n_k < 0.\end{cases}$$

Thus

$$V'_\omega = C, \quad \text{for } \omega \neq 0 \tag{8a}$$

$$V'_\omega = \tfrac{1}{2}(C + C^*) = \text{Re}\{C\}, \quad \text{for } n \neq 0 \text{ and } \omega = 0. \tag{8b}$$

Note that V'_ω is the contribution to $f(i, l, x)$ of one IPD. The two terms in (6) occur as for $n \neq 0$, s_k replaces two sets of q_k and r_k, ($q_k = s_k$, $r_k = s_k + |n_k|$) resulting in the exp $(-j\omega t)$ term, and ($q_k = s_k + |n_k|$, $r_k = s_k$) resulting in the exp $(j\omega t)$ term. For $n = 0$, s_k replaces only one set of q_k and r_k ($q_k = r_k = s_k$). Thus the V'_ω expression is one-half that given in (8b) for the special case $n = 0$.

For (8b) to hold, we need to make the restriction that no IPD (set of n_k's) is equal to the negative of another IPD. The IPD's are only thus affected for $n \neq 0$ and $\omega = 0$. This restriction is necessary to avoid specifying an intermodulation product twice.

If V_ω is the component of y due to a single intermodulation product (as specified by one IPD) then

$$V_\omega = \sum_{i=1}^{I} A_i \sum_{l=0}^{\infty} [a_{l,i} V'_\omega].$$

Defining

$$R_{n+2\sigma,i} = \exp\left(-j\sum_{k=1}^{N} n_k\omega_k \tau_{k,(n+2\sigma),i}\right) = \prod_{k=1}^{N} (\Upsilon'_{k,l,i})^{|n_k|} \tag{9}$$

and introducing the Neumann factor, ϵ_n ($\epsilon_n = 1$, $n = 0$; $\epsilon_n = 2$, $n \neq 0$), we have

$$V_\omega = \text{Re}\left[\epsilon_n\left(\prod_{k=1}^{N} X'^{|n_k|}_k\right)\cdot T\right]_\omega. \tag{10}$$

We define Re $[\;]_\omega$ such that it is ignored for $\omega \neq 0$, but for $\omega = 0$ the real part of the expression in brackets is taken. In (10)

$$T = \sum_{\sigma=0}^{\infty} \frac{(n+2\sigma)!}{2^{(n+2\sigma)}}\cdot z \tag{11}$$

and

$$z = \sum_{\substack{s_1,\cdots,s_N \\ s_1+\cdots+s_N=\sigma}} \left(\prod_{k=1}^{N} \frac{|X_k|^{2s_k}}{s_k!(|n_k|+s_k)!}\right) \cdot \sum_{i=1}^{I}\left(A_i a_{n+2\sigma,i} R_{n+2\sigma,i} \prod_{k=1}^{N}(b_{k,i})^{(|n_k|+2s_k)}\right). \tag{12}$$

Thus the phasor of the ω component of $y(t)$ is given by

$$Y_\omega = \sum_{n=0}^{\infty} \sum_{\substack{n_1,\cdots,n_N \\ |n_1|+\cdots,|n_N|=n}} V_\omega$$

and the n_k's satisfy (5) and the restriction noted previously (no IPD is equal to the negative of another IPD).

That is, the complex amplitude of an output frequency component is the summation of V_ω over all possible IPD's for that frequency.

For the special case of the output equation (1) considered by Sea and Vacroux, (11) and (12) reduce to

$$T = A\sum_{\sigma=0}^{\infty} \frac{a_{n+2\sigma}(n+2\sigma)!}{2^{(n+2\sigma)}}\cdot z$$

$$z = \sum_{\substack{s_1,\cdots,s_N \\ s_1+\cdots+s_N=\sigma}} \prod_{k=1}^{N} \frac{|X_k|^{2s_k}}{s_k!(|n_k|+s_k)!}. \tag{13}$$

Discussion

To analyze a system with noninteracting inputs and outputs, it is only necessary to evaluate the formula once. However, for a system with interacting inputs and outputs, such as a two-terminal nonlinear element with current as input and voltage as output, it is necessary to iterate between the current/voltage solution of the nonlinear element and that of the external linear circuit. When iterating, T tends to change slowly, particularly as convergence is approached. Hence, it need only be calculated every so often, less often as convergence is approached. This can result in up to an order of magnitude reduction in computation time. Note that the summation in i in (12) does not change from one iteration to the next and so need only be calculated initially, thus significantly reducing the computation time required.

The result of Price and Khan [3] can be obtained as a special case of the formula presented here by using the power series coefficients of the expansion of $(1+x)^\alpha$. Price and Khan do not use restriction (9) with the effect that for $n \neq 0$, $\omega = 0$, C of (7) calculated using one IPD is the complex conjugate of that obtained using its negative IPD. Thus (10) becomes their (6).

The result of Sea and Vacroux can also be obtained as a special case. Their V is the magnitude of V_ω, (10). Here the phase of V_ω is obtained

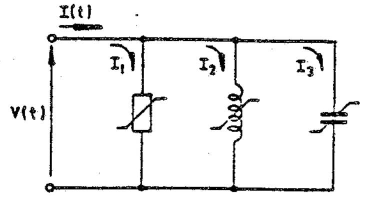

Fig. 1. Circuit with interacting R, L, and C nonlinearities.

as part of the formula, whereas in Sea and Vacroux it is obtained separately [2, eq. (3)].

Example

Here a generalized power series is obtained which describes the terminal I-V characteristics of a circuit with interacting R, L, and C nonlinearities, Fig. 1. The voltage $V(t) = x(t)$, a sum of sinusoids, is the input and the current $I(t) = y(t)$ is the output.

The nonlinear elements are defined by

$$I_1 = \sum_{l=0}^{\infty} G_l V^l(t) \tag{14}$$

$$I_2 = \sum_{l=0}^{\infty} L_l \left(\int V(t)\, dt \right)^l \tag{15}$$

$$I_3 = \frac{d}{dt} \sum_{l=0}^{\infty} C_l V^l. \tag{16}$$

Defining $X_k, Y_\omega, Y_{\omega 1}, Y_{\omega 2}$, and $Y_{\omega 3}$ as the phasors of the frequency components of V, I, I_1, I_2, and I_3 (14)–(16) can be written as

$$Y_{\omega 1} = \sum_{l=0}^{\infty} G_l \left(\sum_{k=1}^{N} X_k(t) \right)^l \tag{17}$$

$$Y_{\omega 2} = \sum_{l=0}^{\infty} L_l \left(\sum_{k=1}^{N} \frac{X_k}{j\omega_k} \right)^l \tag{18}$$

and

$$Y_{\omega 3} = j\omega \sum_{l=0}^{\infty} C_l \left(\sum_{k=1}^{N} X_k \right)^l. \tag{19}$$

Collecting (17)–(19) we obtain a power series of the form of (3) with

$$A_1 = 1 \qquad A_2 = 1 \qquad A_3 = j\omega$$

$$a_{l,1} = G_l \qquad a_{l,2} = L_l \qquad a_{l,3} = C_l$$

$$b_{k,1} = 1 \qquad b_{k,2} = \frac{1}{j\omega_k} \qquad b_{k,3} = 1$$

$$\tau_{k,l,1} = 0 \qquad \tau_{k,l,2} = -\frac{\pi}{2\omega_k} \qquad \tau_{k,l,3} = 0.$$

Conclusion

An algebraic formula was derived for the evaluation of a complex power series, with frequency- and order-dependent time delays, with multifrequency excitation. This formula was related to those obtained by earlier workers.

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