An Algebraic Formula for the Output of a System with Large-Signal, Multifrequency Excitation

MICHAEL B. STEER AND PETER J. KHAN

Abstract—A formula is derived for the output components of a nonlinearity which can be described by a power series, with complex coefficients and frequency-dependent time delays, when the input is a sum of sinusoids.

Introduction

The output $y(t)$ for a system having a multifrequency input $x(t)$, where

$$x(t) = \sum_{k=1}^{N} x_k(t)$$

and

$$x_k(t) = |x_k| \cos (\omega_k t + \phi)$$

has been found by Seaver and Vachon [1], [2] for the case when

$$y(t) = A \sum_{l=0}^{\infty} a_l x(t)^l$$

(1)

and by Price and Khan [3] for

$$y(t) = A (1 + x(t))^\alpha$$

(2)

where $\alpha$ is a noninteger.

Here we consider the general nonlinear system described by a complex power series with frequency-dependent time delays such that

$$y(t) = \sum_{l=1}^{L} A_l \left( \sum_{i=0}^{\infty} a_{l,i} f(i, l, x) \right)$$

(3)

with

$$f(i, l, x) = \left( \sum_{k=1}^{N} b_{k,i} x_k(t - \tau_{k,l,i}) \right)^l$$

and both the $A_l$ and $a_{l,i}$ coefficients are complex, thus indicating a phase shift, while the $b_{k,i}$ coefficients are real. This representation is applicable to a broad class of frequency-dependent nonlinearities, and has recently been used to analyze distortion in microwave FET's [4] and distortion due to phase nonlinearities [5].

Mathematical Development

A component of (3) can be written as

$$x_k(t - \tau_{k,l,i}) = |x_k| \cos (\omega_k t + \phi_k - \omega_k \tau_{k,l,i})$$

$$= \Re \left\{ X_k T_{k,l,i} e^{-j \omega_k t} \right\} = \Re \left\{ X_k T_{k,l,i} e^{-j \omega_k t} \right\}$$

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where \( X_k \) is the phasor of \( x_k \) and
\[
T_{k, l, i} = \exp (-j \omega_k T_{k, l, i}).
\]
\( T_{k, l, i} \) and \( X_k \) are defined separately as this results in a final expression that can be more efficiently computed.
Using the multinomial expansion theorem, we have
\[
f(i, l, x) = \sum_{q_1, q_2, \cdots q_{N_l}, q_1 \cdots q_{N_l} = l \atop q_1 \cdots q_{N_l} = l} \left[ \exp \left( \sum_{k=1}^{N} (q_k - r_k) \omega_k t \right) \right] \prod_{k=1}^{N} \left( \begin{array}{c} N \\ q_k \end{array} \right) (\gamma_k^{q_k} \gamma_k^{r_k} T_{k, l, i}^{q_k} T_{k, l, i}^{r_k}) \frac{q_k!}{r_k!}.
\]
In this equation the frequency of each component is given by
\[
\omega = \sum_{k=1}^{N} n_k \omega_k.
\]
Here the \( n_k \)'s is a set of integers (an intermodulation product description (IPD)) which fixes the difference between \( q_k \) and \( r_k \).
If we let \( s_k \geq 0 \) be the smaller of \( q_k \) and \( r_k \), then \( s_k + |n_k| \) is the larger. Defining
\[
\sigma = \sum_{k=1}^{N} s_k \quad \text{and} \quad n = \sum_{k=1}^{N} n_k |
\]
then for \( n \neq 0 \), one IPD of (4) can be written
\[
(\frac{1}{2} C \exp (j\omega t) + \frac{1}{2} C^* \exp (-j\omega t)) = (\frac{1}{2} V_{\omega} \exp (j\omega t) + \frac{1}{2} V_{\omega}^* \exp (-j\omega t)), \quad \omega \neq 0
\]
\[= V_{\omega}^*, \quad \omega = 0
\]
where
\[
C = \sum_{s_1, s_2, \cdots s_N = \sigma} \prod_{k=1}^{N} \left( \frac{1}{2 s_k + 1} \frac{X_k^{2s_k + 1}(\frac{1}{2} X_k^{s_k})^{n_k}(T_k^{s_k})^{n_k}}{(s_k)! (s_k + |n_k|)!}\right)
\]
and introducing the Neumann factor, \( e_n \) (\( e_n = 1, n = 0; e_n = 2, n \neq 0 \)), we have
\[
V_{\omega} = \text{Re} \left[ e_n \left( \prod_{k=1}^{N} X_k^{n_k} \right) \right] T_{\omega}.
\]
We define \( \text{Re} \left[ _x \right] \) such that it is ignored for \( \omega = 0 \), but for \( \omega = 0 \) the real part of the expression in brackets is taken. In (10)
\[
T = \sum_{\sigma=0}^{\infty} \frac{(n + 2\sigma)!}{2^{n+2\sigma} \cdot \sigma} \cdot z
\]
and
\[z = \sum_{s_1, s_2, \cdots s_N = \sigma} \left( \prod_{k=1}^{N} \frac{|X_k|^{2s_k}}{|s_k! (|n_k| + s_k)!} \right) \cdot \prod_{i=1}^{N} A_{i} e^{n_{i} a_{i} l_{i}} R_{n_{i} a_{i} l_{i}} \prod_{k=1}^{N} (b_k, l, i)^{|n_k| + 2s_k}.
\]
Thus the phase of the \( \omega \) component of \( y(t) \) is given by
\[Y_\omega = \sum_{n=-\infty}^{\infty} \sum_{s_1, s_2, \cdots s_N = \sigma} \frac{X_k^{2s_k}}{|s_k! (|n_k| + s_k)!} \cdot \prod_{i=1}^{N} A_{i} e^{n_{i} a_{i} l_{i}} R_{n_{i} a_{i} l_{i}} \prod_{k=1}^{N} (b_k, l, i)^{|n_k| + 2s_k}.
\]
and the \( n_k \)'s satisfy (5) and the restriction noted previously (no IPD is equal to the negative of another IPD).
That is, the complex amplitude of an output frequency component is the summation of \( V_\omega \) over all possible IPD's for that frequency.
For the special case of the output equation (1) considered by Sea and Varcoux, (11) and (12) reduce to
\[T = A \sum_{\sigma=0}^{\infty} \frac{(n + 2\sigma)!}{2^{n+2\sigma} \cdot \sigma} \cdot z
\]
\[
z = \sum_{s_1, s_2, \cdots s_N = \sigma} \prod_{k=1}^{N} \frac{|X_k|^{2s_k}}{|s_k! (|n_k| + s_k)!}.
\]

**DISCUSSION**

To analyze a system with noninteracting inputs and outputs, it is only necessary to evaluate the formula once. However, for a system with interacting inputs and outputs, such as a two-terminal nonlinear element with current as input and voltage as output, it is necessary to iterate between the current/voltage solution of the nonlinear element and that of the external linear circuit. When iterating, \( T \) tends to change slowly, particularly as convergence is approached. Hence, it need only be calculated every so often, less often as convergence is approached. This can result in a large reduction in computation time. Note that the summation in \( i \) in (12) does not change from one iteration to the next and so need only be calculated initially, thus significantly reducing the computation time required.
The result of Price and Khan [3] can be obtained as a special case of the formula presented here by using the power series coefficients of the expansion of \( (1 + x)^2 \). Price and Khan do not use restriction (9) with the effect that for \( n \neq 0, \omega = 0, C \) of (7) calculated using one IPD is the complex conjugate of that obtained using its negative IPD. Thus (10) becomes their (6).
The result of Sea and Varcoux can also be obtained as a special case. Their \( V \) is the magnitude of \( V_\omega \), (10). Here the phase of \( V_\omega \) is obtained.
as part of the formula, whereas in Sea and Vacroux it is obtained separately [2, eq. (3)].

**Example**

Here a generalized power series is obtained which describes the terminal $I-V$ characteristics of a circuit with interacting $R$, $L$, and $C$ nonlinearities, Fig. 1. The voltage $V(t) = x(t)$, a sum of sinusoids, is the input and the current $I(t) = y(t)$ is the output.

The nonlinear elements are defined by

$$I_1 = \sum_{l=0}^{\infty} G_i V^l(t)$$

(14)

$$I_2 = \sum_{l=0}^{\infty} L_l \left( \int V(t) \, dt \right)^l$$

(15)

$$I_3 = \frac{d}{dt} \sum_{l=0}^{\infty} C_l V^l.$$  

(16)

Defining $X_k$, $Y_1, Y_2, Y_3$, and $Y_3$ as the phasors of the frequency components of $V, I, I_1, I_2$, and $I_3$ (14)-(16) can be written as

$$Y_1 = \sum_{k=1}^{N} G_i \left( \sum_{k=1}^{N} X_k \right)^l$$

(17)

$$Y_2 = \sum_{l=0}^{\infty} \frac{L_l}{l!} \left( \sum_{k=1}^{N} X_k \right)^l$$

(18)

and

$$Y_3 = j\omega \sum_{l=0}^{\infty} \frac{C_l}{C_l} \left( \sum_{k=1}^{N} X_k \right)^l.$$  

(19)

Collecting (17)-(19) we obtain a power series of the form of (3) with

$$A_1 = 1 \quad A_2 = 1 \quad A_3 = j\omega$$

$$a_{1,1} = G_i \quad a_{1,2} = L_l \quad a_{1,3} = C_i$$

$$b_{k,1} = 1 \quad b_{k,2} = \frac{1}{j\omega_k} \quad b_{k,3} = 1$$

$$\tau_{k,1} = 0 \quad \tau_{k,2} = -\frac{\pi}{2\omega_k} \quad \tau_{k,3} = 0.$$  

**Conclusion**

An algebraic formula was derived for the evaluation of a complex power series, with frequency- and order-dependent time delays, with multifrequency excitation. This formula was related to those obtained by earlier workers.

**References**

