Frequency-Domain Nonlinear Circuit Analysis Using Generalized Power Series

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Abstract —**This paper presents for the first time details of the generalized power series technique for the analysis of analog nonlinear circuits. The method uses generalized power series descriptions of the nonlinear elements and a spectral balance technique to operate entirely in the frequency domain. It is therefore suited to the analysis of analog nonlinear circuits with large-signal multifrequency excitation of arbitrary frequency separation. The analysis of a low-frequency mixer is used here as a vehicle to illustrate the concepts of large-signal frequency-domain analysis and the generalized power series analysis technique.**

I. INTRODUCTION

THE TREND TOWARD monolithic integration of microwave circuits is intensifying interest in computer-aided design. The interest is predominantly in the analysis and design of microwave nonlinear analog circuits having sinusoidal excitation. The particular problems presented by these circuits differ significantly from those of low-frequency and of digital circuits and require new simulation strategies. Currently, research is proceeding in several areas, including large-signal multifrequency excitation (including mixer and intermodulation analysis), optimization, noise analysis, and stability analysis.

This paper details a recently developed numerical nonlinear analysis technique that can be used with analog circuits having large-signal multifrequency excitation. Modified power series descriptions (having time delays and complex coefficients) of the nonlinear elements are used, so we term the method *generalized power series analysis* (GPSA). Earlier uses of generalized power series analysis are described in [1]–[3]. Applications of generalized power series analysis are reported elsewhere for the simulation of microwave mixers [4], [5] and of IMPATT oscillators [6]–[8]. More recently, GPSA has been used to simulate gain compression and intermodulation distortion in microwave MESFET amplifiers [9]–[11]. In each application, the predictions of generalized power series analysis have been experimentally verified. In this paper, the method of generalized power series analysis is elaborated for the first time. We present the basic equations dealing with generalized power series and show how they can be incorporated into a harmonic-balance-type algorithm, including the efficient calculation of the Jacobian matrix. We discuss the application of the technique to the analysis of a low-frequency mixer to illustrate the concepts of large-signal frequency-domain analysis and GPSA.

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II. METHODS OF MULTIFREQUENCY NONLINEAR ANALYSIS

Nonlinear analysis methods can be classified as time-domain, frequency-domain, or hybrid (mixed time- and frequency-domain) methods depending on how the linear and nonlinear elements are analyzed. Time-domain methods generally use numerical integration or, where possible, calculate the instantaneous value of the output (e.g. current) of an element from the instantaneous value of the input (e.g. voltage) to it. An example of a computer-aided analysis technique using this approach is the popular computer program SPICE [12]. Microwave circuits often have elements that are difficult to model in the time domain and frequently have time constants that differ by orders of magnitude. Analysis of these circuits using numerical integration techniques is inefficient since the integration time step must be smaller than twice the smallest time constant while the number of iterations required is determined by the largest time constant [15]. Analysis of circuits having multifrequency excitation is similarly troublesome, particularly when the frequencies considered are widely separated. Time-domain methods also suffer from poor dynamic range, which is a problem whenever signals having large differences in amplitude are present, a situation common in mixer and amplifier circuits.

A far more useful technique for analyzing microwave circuits is the harmonic balance method [16]. This method analyzes the linear elements in the frequency domain and the nonlinear elements in the time domain and has successfully been applied to microwave circuit analysis by a number of researchers [17]–[21]. With these analyses, the conversion between the frequency-domain solution of the linear embedding network and the time-domain solution of the nonlinearity is usually accomplished using fast Fourier

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transform techniques. This limits these methods to systems having only harmonically related signal components. Recently, however, several methods have been described which allow consideration of more general multifrequency excitation. Ushida and Chua [22] have developed a technique that uses a "generalized" discrete Fourier transform, which has been improved by Sorkin *et al.* [23]. Gilmore and Rosenbaum [24] achieve multifrequency harmonic-balance simulation by sampling the time-domain signal below the Nyquist rate. The actual response is found by shifting the input spectra, repeating the analysis, and appropriately combining the resulting output spectra. A more straightforward approach has been used by Rizzoli *et al.* [25], where a large number of harmonics are considered and a supercomputer used for computation. While these methods are frequently classified as frequency-domain techniques, they are more appropriately called hybrid methods since much of the analysis is explicitly done in the time domain.

Yet another approach to circuit analysis is to treat both the linear and the nonlinear elements in the frequency domain. This has particular advantages when the frequency-domain response of a nonlinear analog circuit is required. Frequency-domain nonlinear analyses use functional expansions of the input–output characteristic of the nonlinear element. Generally, the function itself is the summation of basis functions, and the responses due to each functional component of the expansion are summed to yield the total response of the system. Perhaps the most general method is that of Antonov and Ponkratov [26], who derived a formula for the output of a system described by the functional relation $y(t)=F(x(t))$, where $x(t)$ is a sum of sinusoids, and $F(\cdot)$ is a function which can be expressed as a possibly infinite sum of orthonormal functions. Their output formula involved multiple infinite summations of integer order Bessel functions. The result of using Bessel functions is that convergence of the summations is likely to be slow and to suffer from poor numerical accuracy.

Two other frequency-domain nonlinear analysis methods, using power series and Volterra series, can be viewed as special cases of the system described by Antonov and Ponkratov. The simplest functional expansion is the representation of $y(t)$ as a power series in $x(t)$. Conventional power series expansions can only be used with frequency-independent (i.e., resistive) systems having single valued input–output characteristics (i.e., without hysteresis) [13], [14]. Other basis function expansions, such as the expansion of the Shockley diode equation in terms of Bessel functions [27], [28], have been used but these are generally restricted to systems with particular idealized input–output characteristics.

In 1930 Volterra introduced functional expansions that could be used with a large class of nonlinear systems. His work was developed further by Weiner in the 1950's for the expansion of functionals in terms of orthogonal polynomial series. Weiner's functional expansions, now known as Volterra nonlinear transfer functions, while having a form similar to that of a power series, can handle frequency-dependent systems with single valued input–output characteristics [29]. Unfortunately, Volterra nonlinear transfer function analyses are, in general, restricted to weakly nonlinear systems because of the algebraic complexity of determining Volterra nonlinear transfer functions of high order (as required with more strongly nonlinear systems or with large signals). Because of this, systems are usually described by fixed, typically third-order, Volterra series, although no indication of the error involved in doing this is available The limitation arises as, in essence, Volterra series analysis involves an algebraic process analogous to recursion of power series. This is an unwieldy operation and is exceedingly complex for Volterra series higher than third order. The great importance of Volterra series analysis is that it can be systematically used to analyze fairly complex systems with possibly noncommensurable frequencies of the input components. These techniques have been used successfully in analyzing microwave circuits [30], [31]. An approach related to Volterra series expansion has been recently reported by Lamnabhi [32]. This method has the advantage of being more amenable to computer implementation than the traditional approaches.

Yet another frequency-domain nonlinear analysis was introduced by Steer and Khan [1], [4], who used a generalized power series expansion of the input–output characteristics of a nonlinear system. This method is related to Volterra series analysis [33]; however, the generalized power series method is not restricted to weakly nonlinear systems, as is Volterra series analysis. The basic properties of generalized power series are reviewed in the following section.

III. Basic Properties of Generalized Power Series

Every nonlinear circuit simulator must have a facility for calculating the output of a nonlinearity given the input and a description of the nonlinear element. The method of calculation selected determines the types of problems that can be efficiently solved. If a transient response is required, for example, then a time-domain simulation is indicated. If instead, the circuit is excited by a periodic signal and only harmonic frequency components are present, then a harmonic balance simulation will be more efficient. If, however, several signals are present that are not harmonically related, then a simulator that can directly calculate the output given the multifrequency input could be the most efficient. This is the motivation for using GPSA. Here, with the input (e.g. voltage) being a sum of sinusoids having arbitrary frequency relationships, the output (e.g. current) at each frequency is calculated independently. The formulas used in this calculation are detailed below.

In the method of Steer and Khan the output $y(t)$ of a system having an N-component multifrequency input

$$x(t)=\sum_{k=1}^{N} x_k(t)=\sum_{k=1}^{N}|X_k|\cos(\omega_k t+\phi_k) \qquad (1)$$

is described by the generalized power series

$$y(t)=A\sum_{l=0}^{\infty}\left[a_l\left\{\sum_{k=1}^{N}b_k x_k(t-\tau_{k,l})\right\}^l\right]. \quad (2)$$

Here $y(t)$ is the output of the system; l is the order of the power seriesterms; a_l is a complex coefficient; $\tau_{k,l}$ is a time delay that depends on both power series order and the index of the input frequency component; and b_k is a real coefficient. Using complex coefficients and time delays enables a broad class of nonlinear circuits and systems to be described by generalized power series [1]–[4], [6]–[11]. Note that $|X_k|$ is the peak magnitude of an input sinusoid so that a dc input component has $\omega_k=0$ and $\phi_k=0$ or π radians. In phasor notation,

$$\begin{aligned}x_k(t-\tau_{k,l})&=|X_k|\cos(\omega_k t+\phi_k-\omega_k\tau_{k,l})\\&=\tfrac{1}{2}X_k\Gamma_{k,l}e^{j\omega_k t}+\tfrac{1}{2}X_k^*\Gamma_{k,l}^*e^{-j\omega_k t}\end{aligned}$$

where X_k is the phasor of x_k and

$$\Gamma_{k,l}=\exp(-j\omega_k\tau_{k,l}).$$

Using the multinomial expansion theorem, the power series of (2) can be expanded and terms collected according to frequency. As a result, the phasor component of the output, Y_q, corresponding to the radian frequency ω_q, can be expressed as a sum of intermodulation products (various powers of X_k multiplied together) as given in [1]

$$Y_q=\sum_{n=0}^{\infty}\sum_{\substack{n_1,\cdots,n_N\\|n_1|+\cdots+|n_N|=n}}U_q \quad (3)$$

where $\omega_q=\sum_{k=1}^{N}n_k\omega_k$, a set of n_k's defines an intermodulation product (called an IPD), and n is the order of intermodulation. The second summation is over all possible combinations of $n_1,\cdots,n_N$ such that $|n_1|+\cdots+|n_N|=n$. The summations are therefore over the infinite number of intermodulation products (the U_q's) yielding the qth output component (Y_q). When a nonlinear circuit is excited by a finite number of sinusoids, an infinite number of frequency components are present. In order to analyze such a problem numerically, the number of frequency components considered in the analysis must be truncated. Here we consider N frequency components. Each intermodulation product in (3) is found from

$$U_q=\mathrm{Re}\left\{A\epsilon_n\left(\prod_{k=1}^{N}(X_k')^{|n_k|}\right)T\right\}_{\omega_q} \quad (4)$$

where

$$T=\sum_{\sigma=0}^{\infty}\left\{\left(\frac{(n+2\sigma)!}{2^{(n+2\sigma)}}\right)a_{n+2\sigma}R_{n+2\sigma}Z\right\} \quad (5)$$

and

$$Z=\sum_{\substack{s_1,\cdots,s_N\\s_1+\cdots+s_N=\sigma}}\left\{\left(\prod_{k=1}^{N}\frac{|X_k|^{2s_k}}{s_k!(|n_k|+s_k)!}\right)\left(\prod_{k=1}^{N}b_k^{(|n_k|+2s_k)}\right)\right\}. \quad (6)$$

In these expressions X_k is the phasor of x_k,

$$X_k'=\begin{cases}X_k & n_k\geq 0\\ X_k^* & n_k<0\end{cases} \quad (7)$$

$$R_{n+2\sigma}=\exp\left(-j\sum_{k=1}^{N}n_k\omega_k\tau_{k,n+2\sigma}\right) \quad (8)$$

$$\epsilon_n=\begin{cases}1 & n=0\\ 2 & n\neq 0\end{cases} \quad (9)$$

and $\mathrm{Re}\{\ \}_{\omega_q}$ is defined such that for $\omega_q\neq 0$ it is ignored and for $\omega_q=0$ the real part of the expression in braces is taken. The formula (3)–(9) essentially turns a time-domain description (the generalized power series) into a frequency-domain description (the algebraic formula). GPSA has the advantage of retaining the time-domain description of the nonlinearities but requires no explicit time-domain calculations in order to calculate the frequency-domain representation for the output. The formula is considerably simpler for nonlinear components that can be described by conventional power series [1].

In addition, formulas can be derived for calculating the derivatives of the output phasors with respect to the input quantities. Partial derivatives of a nonlinear node current phasor with respect to the magnitude and phase of a node voltage phasor are obtained by differentiating the algebraic formula (3)–(9). Using the notation in (1)–(9), the derivative of the phasor of the qth component of the output of the nonlinearity with respect to the magnitude of the phasor of the mth input component, $X_m=|X_m|e^{j\phi_m}$, is found from (3) by differentiating

$$\frac{\partial Y_q}{\partial|X_m|}=\sum_{n=0}^{\infty}\sum_{\substack{n_1,\cdots,n_N\\|n_1|+\cdots+|n_N|=n}}\frac{\partial U_q}{\partial|X_m|} \quad (10)$$

where

$$\begin{aligned}\frac{\partial U_q}{\partial|X_m|}=&\frac{|n_m|}{|X_m|}U_q+A\epsilon_n\left(\prod_{k=1}^{N}(X_k')^{|n_k|}\right)\\&\cdot\left(\sum_{\sigma=0}^{\infty}a_{n+2\sigma}R_{n+2\sigma}\frac{(n+2\sigma)!}{2^{(n+2\sigma)}}\frac{\partial Z}{\partial|X_m|}\right)\end{aligned} \quad (11)$$

and

$$\begin{aligned}\frac{\partial Z}{\partial|X_m|}=&\sum_{\substack{s_1,\cdots,s_N\\s_1+\cdots+s_N=\sigma}}\left\{\left(\prod_{\substack{k=1\\k\neq m}}^{N}\frac{|X_k|^{2s_k}}{s_k!(|n_k|+s_k)!}\right)\right.\\&\left.\cdot\left(\frac{2s_m|X_m|^{2s_m-1}}{s_m!(|n_m|+s_m)!}\right)\left(\prod_{k=1}^{N}b_k^{(|n_k|+2s_k)}\right)\right\}.\end{aligned} \quad (12)$$

Similarly, the derivative of the phasor of the qth component of the output of the nonlinearity with respect to the angle of the phasor of the mth component of the input is found to be

$$\frac{\partial Y_q}{\partial\phi_m}=\sum_{n=0}^{\infty}\sum_{\substack{n_1,\cdots,n_N\\|n_1|+\cdots+|n_N|=n}}\frac{\partial U_q}{\partial\phi_m} \quad (13)$$

where

$$\frac{\partial \boldsymbol{U}_q}{\partial \phi_m} = jn_m \boldsymbol{U}_q. \tag{14}$$

Calculation of the partial derivatives is, relatively, computationally inexpensive, as many of the terms are precalculated in the evaluation of an intermodulation product. The following section will show how these formulas can be incorporated in a harmonic-balance-type algorithm to analyze nonlinear circuits.

IV. Generalized Power Series and Spectral Balance

The analysis method presented in this paper is based on minimization of an objective function derived from the application of Kirchhoff's current law. The nonlinear elements are described using generalized power series while the linear elements are handled using standard frequency-domain nodal techniques. This results in an efficient analysis procedure, which is described below. We show how the objective function is calculated and then present an efficient method for minimizing the objective function as well as an algorithm for implementing the analysis technique on a digital computer.

The analysis of a nonlinear circuit proceeds by dividing the circuit into linear and nonlinear subcircuits as shown in Fig. 1. One subcircuit is composed of the linear components along with any voltage or current sources, while the other is composed of nonlinear elements, each of which is characterized by a generalized power series. The nonlinear subcircuit has M nodes and at the pth node the instantaneous current into the linear subcircuit is the sum of N frequency components so that

$$i_p = \sum_{q=1}^{N} \operatorname{Re}\left[\boldsymbol{I}_{p,q} e^{j\omega_q t}\right]. \tag{15}$$

Similarly, the current into the nonlinear subcircuit at the pth node is

$$i'_p = \sum_{q=1}^{N} \operatorname{Re}\left[\boldsymbol{I}'_{p,q} e^{j\omega_q t}\right] \tag{16}$$

where $\boldsymbol{I}_{p,q}$ and $\boldsymbol{I}'_{p,q}$ are the phasors of the qth frequency components of current flowing into the linear and nonlinear subcircuits, respectively. The voltage at the pth node is

$$v_p = \sum_{q=1}^{N} \operatorname{Re}\left[\boldsymbol{V}_{p,q} e^{j\omega_q t}\right] \tag{17}$$

where $\boldsymbol{V}_{p,q}$ is the phasor of the qth frequency component of voltage at the pth node (referred to as a node voltage phasor). Kirchhoff's current law must be satisfied, so $i_p + i'_p = 0$ for all p from 1 to M. Thus, the steady-state solution of the circuit can be found by minimizing the objective function

$$E = \sum_{p=1}^{M} \sum_{q=1}^{N} |\boldsymbol{I}_{p,q} + \boldsymbol{I}'_{p,q}|^2. \tag{18}$$

For efficient computation, the objective function E is

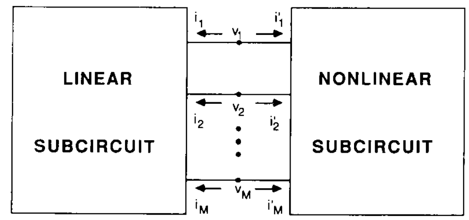


Fig. 1. A nonlinear circuit separated into linear and nonlinear subcircuits. The instantaneous current into the linear subcircuit at the pth node is i_p, while i'_p flows into the nonlinear subcircuit. The instantaneous voltage at the node is v_p.

written as

$$E = \sum_{j=1}^{2} \sum_{p=1}^{M} \sum_{q=1}^{N} F_{j,p,q}^2(\boldsymbol{V}) = \sum_{i=1}^{2MN} G_i^2(\boldsymbol{V}) \tag{19}$$

where

$$F_{1,p,q}(\boldsymbol{V}) = \operatorname{Re}(\boldsymbol{I}_{p,q} + \boldsymbol{I}'_{p,q}) \tag{20}$$

and

$$F_{2,p,q}(\boldsymbol{V}) = \operatorname{Im}(\boldsymbol{I}_{p,q} + \boldsymbol{I}'_{p,q}). \tag{21}$$

The elements $G_i(\boldsymbol{V})$ are equal to the elements $F_{j,p,q}$, where the subscript i represents a unique choice for j, p, and q. In these expressions, $\boldsymbol{V}$ is the vector of the node voltage phasors. Evaluation of the objective function as a function of the node voltage phasors requires calculation of the node current phasors given the node voltage phasors. For the linear subcircuit the node current phasors are easily calculated using standard frequency-domain nodal techniques whereas the current in the nonlinear subcircuit can be calculated using the algebraic formula (3)–(9) since the nonlinear elements are described by generalized power series.

Minimization of the objective function can be achieved using a variety of iteration schemes. One suitable technique to minimize such a sum of squares is Newton's method. This method seeks to find the minimum of E with respect to $\boldsymbol{V}$ using the iterative procedure

$$^{i+1}\begin{bmatrix} \operatorname{Re}(\boldsymbol{V}_{1,1}) \\ \operatorname{Im}(\boldsymbol{V}_{1,1}) \\ \vdots \\ \operatorname{Re}(\boldsymbol{V}_{p,q}) \\ \operatorname{Im}(\boldsymbol{V}_{p,q}) \\ \vdots \\ \operatorname{Re}(\boldsymbol{V}_{M,N}) \\ \operatorname{Im}(\boldsymbol{V}_{M,N}) \end{bmatrix} = {}^{i}\begin{bmatrix} \operatorname{Re}(\boldsymbol{V}_{1,1}) \\ \operatorname{Im}(\boldsymbol{V}_{1,1}) \\ \vdots \\ \operatorname{Re}(\boldsymbol{V}_{p,q}) \\ \operatorname{Im}(\boldsymbol{V}_{p,q}) \\ \vdots \\ \operatorname{Re}(\boldsymbol{V}_{M,N}) \\ \operatorname{Im}(\boldsymbol{V}_{M,N}) \end{bmatrix} - \boldsymbol{J}^{-1}({}^{i}\boldsymbol{V})\boldsymbol{G}({}^{i}\boldsymbol{V}) \tag{22}$$

where the leading superscripts are iteration numbers. The matrix $\boldsymbol{J}$ is the Jacobian where the element in the $(2j-1)$th row and kth column at the ith iteration is

$$\left[\boldsymbol{J}({}^{i}\boldsymbol{V})\right]_{2j-1,k} = \frac{\partial G_{2j-1}({}^{i}\boldsymbol{V})}{\partial \operatorname{Re}({}^{i}\boldsymbol{V}_k)} \tag{23}$$

and at the $(2j)$th row and kth column

$$\left[J({}^iV)\right]_{2j,k} = \frac{\partial G_{2j}({}^iV)}{\partial \operatorname{Im}\left({}^iV_k\right)}. \tag{24}$$

Calculation of the Jacobian requires partial derivatives of the node current phasors with respect to the real and imaginary parts of the node voltage phasors for all nodes of the nonlinear subcircuit and frequency components. These derivatives are readily available for the linear subcircuit, and can be calculated for the nonlinear subcircuit when the elements are described by generalized power series. In fact, one of the major advantages of generalized power series analysis is that the partial derivatives can be readily and efficiently calculated.

While the derivatives calculated in (10)–(14) are with respect to polar quantities, they are easily converted to derivatives with respect to the real and imaginary components of the node voltage phasors by using the chain rule as follows:

$$\frac{\partial Y_q}{\partial \operatorname{Re}(X_m)} = \frac{\partial Y_q}{\partial |X_m|}\frac{\partial |X_m|}{\partial \operatorname{Re}(X_m)} + \frac{\partial Y_q}{\partial \phi_m}\frac{\partial \phi_m}{\partial \operatorname{Re}(X_m)} \tag{25}$$

and

$$\frac{\partial Y_q}{\partial \operatorname{Im}(X_m)} = \frac{\partial Y_q}{\partial |X_m|}\frac{\partial |X_m|}{\partial \operatorname{Im}(X_m)} + \frac{\partial Y_q}{\partial \phi_m}\frac{\partial \phi_m}{\partial \operatorname{Im}(X_m)}. \tag{26}$$

The derivatives for the linear subcircuit are available as the y parameters of the subcircuit. The derivative of the current through the linear admittance (Y) with respect to the real component of the phasor voltage across it is

$$\frac{\partial I_q}{\partial \operatorname{Re}(V_m)} = \begin{cases} Y & m = q \\ 0 & m \neq q \end{cases} \tag{27}$$

while the derivative with respect to the imaginary component is

$$\frac{\partial I_q}{\partial \operatorname{Im}(V_m)} = \begin{cases} jY & m = q \\ 0 & m \neq q \end{cases}. \tag{28}$$

The simulation technique just described is implemented in the program FREDA (FREquency Domain Analysis) [11], which uses the algorithm outlined in Fig. 2. The analysis of a nonlinear circuit proceeds as follows. Initially the circuit and device parameters are input and the y parameters of the linear subcircuit are calculated at all frequencies. Then an initial estimate of the node voltages is used to calculate the currents in the circuit, along with the objective function and the necessary derivatives. The initial estimate of the node voltages need not be very precise and a zero initial guess is usually adequate. The magnitude of the objective function calculated from the initial voltage estimate is checked and if it is sufficiently small, the voltage estimate is taken as the steady-state solution. Otherwise, the voltage estimate is updated using the iteration procedure (22). These steps are repeated until the steady-state solution is found. It should be noted that the method just developed requires that the nonlinear elements be described using a series of the form (2), e.g. a generalized power series in one variable. Work is in progress to extend these techniques to functions of several variables by generalizing the algebraic formula of equations (3)–(9).

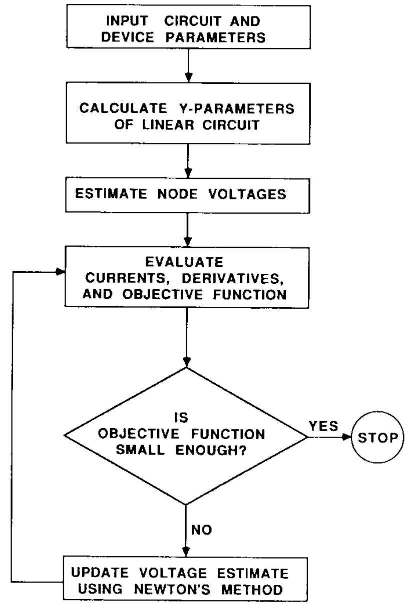


Fig. 2. Algorithm for the generalized power series nonlinear circuit analysis using a minimization procedure as used in the program FREDA.

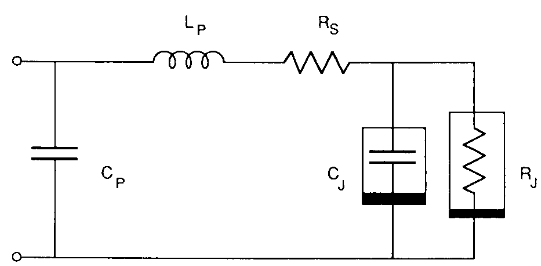


Fig. 3. Equivalent circuit model of a pn junction or Schottky barrier diode.

V. Example: Low-Frequency Mixer Analysis

Mixers couple, via a nonlinear device, a large local oscillator signal (LO) and an input signal (RF). The signals mix to produce components at sum and difference frequencies, with one extracted as the desired output signal and termed the intermediate frequency (IF). As the nonlinear device, mixers commonly use a pn junction or a Schottky barrier diode, modeled by a nonlinear resistance in parallel with a nonlinear capacitance (Fig. 3). The traditional approach to mixer analysis is first to neglect the RF and solve for the LO waveform at the diode junction. This is performed in the time domain as the waveform at the diode is then periodic and its frequency components can be determined using a fast Fourier transform. Next a small-signal frequency-domain conversion matrix is developed to relate the RF, IF, and LO sidebands. In the terminology used in this paper, this conversion matrix describes the low-order intermodulation present when the RF is negligibly small. The conversion matrix concept can

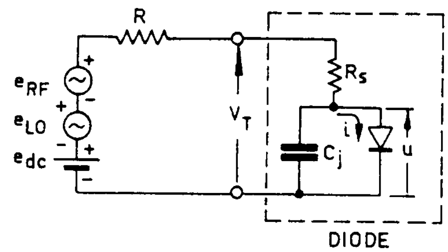


Fig. 4. Resistive mixer circuit with a general-purpose small-signal Si pn junction diode, 1N4148. dc bias $V_d = 1$ V, peak LO voltage $V_p = 1.4$ V, peak RF voltage $V_s = 0.1$ V, $R = 1000\ \Omega$, $R_s = 7\ \Omega$, $C_j = 1\ \mu$F. The LO frequency is 1.1 kHz and the input signal is 1.3 kHz, yielding an intermediate frequency of 200 Hz, $i = I_s\{\exp(qv/\eta kT) - 1\}$, $I_s = 3.16$ nA and $\eta = 1.895$, at 22°C. The ideal series connection of the voltage generators was achieved using an operational amplifier in the summing mode. The voltage across the terminals of the diode model, v_T, was measured through a unity-gain buffer amplifier to eliminate parasitic effects.

be extended to include a large RF by considering that the extra intermodulation products that result add a dimension to the conversion matrix. GPSA calculates all significant intermodulation products and could be used to develop this extended, although signal-level-dependent, conversion matrix. However it is not necessary to develop the "extended" conversion matrix explicitly as the intermodulation products can be used directly to determine mixer response as is done here.

Large-signal analysis of purely resistive diode mixers has been based on the Sonine expansion of the ideal Shockley diode equation [28]. Whereas generalized power series analysis can be used with arbitrary single-valued junction characteristics, the analysis based on the Sonine expansion can only be used with an exponential resistive nonlinearity. The Sonine expansion and generalized power series methods can therefore only be compared directly at low frequencies, when the nonlinear capacitance of a diode is negligible. Using the low-frequency mixer of Fig. 4, we compare experimental results and simulations based on the Sonine expansion analysis and GPSA. Low-frequency, rather than microwave-frequency, experimentation is more reliable as the mixer can be accurately characterized and measurement parasitics eliminated.

A. Development of Generalized Power Series

FREDA uses a spectral balance iteration scheme, iterating between the current–voltage solution of the nonlinear element and that of the linear circuit to minimize the Kirchhoff's current law error. Thus the generalized power series describing the current–voltage relationship of the nonlinear element must be developed. This can be derived from the ideal algebraic relations or from experimental measurements using a curve fitting procedure so that nonideal effects such as high injection and diffusion capacitance effects can be incorporated. It is not necessary to model the nonlinearity by lumped elements. By way of example, the generalized power series description is developed here from the ideal current–voltage and capacitance–voltage equations that approximately model the active region of the diode.

The form of the generalized power series can significantly affect computation time. Up to several orders of magnitude increase in computation speed can be obtained by developing power series in alternating components only (that is, the dc value or operating point is not included in the generalized power series description). This is because in most circuits and systems dc is the dominant component but it does not directly contribute to the intermodulation phenomena. The generalized power series development below results in a power series that separates dc and ac quantities; however, this need not be done explicitly as it can be performed automatically.

The current–voltage characteristic of an ideal pn junction or Schottky barrier diode is described by the Schockley diode equation

$$i_R = I_s[\exp(\alpha v) - 1] \tag{29}$$

and by the capacitance–voltage characteristic

$$C_j = \frac{C_{j0}}{\left(1 - \dfrac{v}{\phi}\right)^{\gamma}} \tag{30}$$

where i_R is the current into the nonlinear resistor, v is the voltage across the nonlinear resistor and nonlinear capacitor, C_j is the nonlinear junction depletion capacitance, and the other parameters are constants. In the following development the direct component of voltage and current has the subscript $k = 1$, and alternating components have $k = 2, \cdots, N$. If the voltage across the diode consists of direct, $V_d = v_1$, and alternating, $v_a = \Sigma_{k=2}^{N} v_k$, terms such that $v = V_d + v_a$ (here and in the following the subscripts d and a denote direct and alternating quantities, respectively), then the Taylor series expansion of (29) yields

$$i_R = I_s \exp(\alpha V_d)\left\{\sum_{l=0}^{\infty}\left(\frac{\alpha^l}{l!}v_a^l\right)\right\} - I_s. \tag{31}$$

The expressions describing the nonlinear capacitor can be similarly developed. Under steady-state conditions, the average current into the nonlinear capacitor must be zero since, for a lossless capacitor charge cannot pass between the plates. Thus it is only necessary to obtain the generalized power series for the alternating components of current. Integrating both sides of (30) and setting the depletion layer charge $q = 0$ when $v = \phi$ [34], the charge is given by

$$q = \frac{C_{j0}\phi^{\gamma}}{\gamma - 1}(\phi - v)^{1-\gamma} \tag{32}$$

$$= \frac{C_{j0}\phi^{\gamma}(\phi - V_d)^{1-\gamma}}{\gamma - 1}\sum_{l=0}^{\infty} a_l \left\{1 - \frac{v_a}{\phi - V_d}\right\}^l \tag{33}$$

where

$$a_l = \begin{cases} \dfrac{(1-\gamma)(-\gamma)\cdots(1-\gamma-l+1)}{l!} & l = 1, 2, \cdots \\ 1 & l = 0. \end{cases} \tag{34}$$

If I_k and Q_k are phasor components, with radian frequency ω_k, of current and charge, then $I_k = j\omega_k Q_k$. Thus the current through the capacitor can be expressed in terms of a generalized power series of voltage

$$i_C = \frac{j\omega C_{j0}\phi^{\gamma}(\phi - V_d)^{1-\gamma}}{\gamma - 1} \sum_{l=0}^{\infty} a_l \left\{ \frac{-v_a}{\phi - V_d} \right\}^l. \quad (35)$$

This is not a conventional power series insofar as ω, the radian frequency of a component of i, is not unique. Equation (35) also applies to the direct current through C_j, as then $\omega = 0$ and the dc component calculated by (35) is zero.

The generalized power series expressions for i_C and i_R can be combined to produce a single generalized power series for the total current through the diode junction:

$$i = \sum_{m=1}^{2} A_m \left\{ \sum_{l=0}^{\infty} a_{l,m} \left\{ \sum_{k=2}^{N} b_{k,m} v_k(t) \right\}^l \right\} \quad (36)$$

where

$$A_1 = I_s \exp(\alpha V_d)$$

$$a_{l,1} = \frac{\alpha^l}{l!}$$

$$b_{k,1} = 1$$

$$A_2 = \frac{j\omega C_{j0}\phi^{\gamma}(\phi - V_d)^{1-\gamma}}{\gamma - 1}$$

$$a_{l,2} = \begin{cases} \dfrac{(1-\gamma)(-\gamma)\cdots(1-\gamma-l+1)}{l!} & l = 1, 2, \cdots \\ 1 & l = 0 \end{cases}$$

$$b_{k,2} = \frac{-1}{\phi - V_d}.$$

This generalized power series is input to the program FREDA. Of course it need not be so complicated to develop a generalized power series description of a nonlinear element. The above served to illustrate how a generalized power series can be developed from analytic expressions and in so doing leads to an understanding of the generalized power series. In this case, use of analytic expressions will result in a power series that has a limited range of convergence. In practice, a power series can be found to accurately model the nonlinearity over a sufficiently wide range of input voltage using numerical methods.

B. Generalized Power Series Analysis

At each iteration, FREDA uses the *algebraic formula*, (3)–(9), to evaluate individual intermodulation products that are summed to yield an estimate of the response of a nonlinear system. The iteration scheme described previously then determines the actual response of the system to a prescribed accuracy. The third (after formula evaluation and iteration scheme) major aspect of GPSA is determining the intermodulation products to calculate.

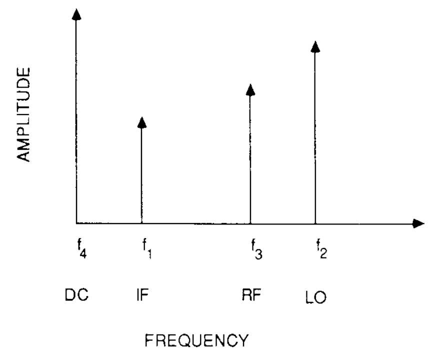


Fig. 5. Simplified spectrum at the terminals of the diode in a mixer.

Each intermodulation product is defined by an intermodulation product description (IPD)—a set of n_k's so that the frequency of an intermodulation product is given by $\omega = \sum_{k=1}^{N} n_k \omega_k$ and $n = \sum_k |n_k|$ is the order of intermodulation. IPD's up to a maximum order $n_{\max}$ are predetermined and stored in a data base although in the evaluation of the formula, all intermodulation products of the same order are calculated and added to the total response for that frequency component until the desired fractional accuracy is obtained.

Here we use the simplified spectrum (Fig. 5), of the waveform at the mixer diode to illustrate the concept of intermodulation products and IPD's. Fig. 5 is a simple spectrum of the waveform at the diode and retains only those components integral to the operation of a mixer: f_2 is the LO; f_3 is the RF; f_4 is dc; and $f_1 = f_2 - f_3$ is the IF. All of these components appear at the terminals of the diode junction and so all (e.g. voltage) components are *inputs* to the algebraic formula. A partial listing of the intermodulation products is given in Table I. For example, Table I lists the fourth-order intermodulation product description $f_1 = 2f_1 - f_2 + f_3$, which yields a component at f_1 and contains an intermodulation product of the form $X_1^2 X_2^* X_3$. The evaluation of the *algebraic formula* (3)–(9) for all IPD's is illustrated in Fig. 6. This shows that each intermodulation product is calculated independently and summed to give the output at a particular frequency.

The experimental and calculated voltage v_T at the terminals of the diode model is given in Fig. 7. Using the generalized power series analysis, the diode waveform spectrum, indicated by +'s in Fig. 7(b), was calculated. The eight frequency components used in the analysis were chosen to include the components with the most influence in determining the behavior being examined. With mixers, the conversion loss is the most important parameter to be found and so the dc, lower-order LO harmonics, RF, IF, and low-order sidebands of the LO harmonics must be included in the analysis. The measured conversion loss (defined here as the ratio of the IF power delivered by the diode and the power supplied by the input signal generator) $L_c = 12.5$ dB compares favorably to the calculated conversion loss $L_c = 11.5$ dB.

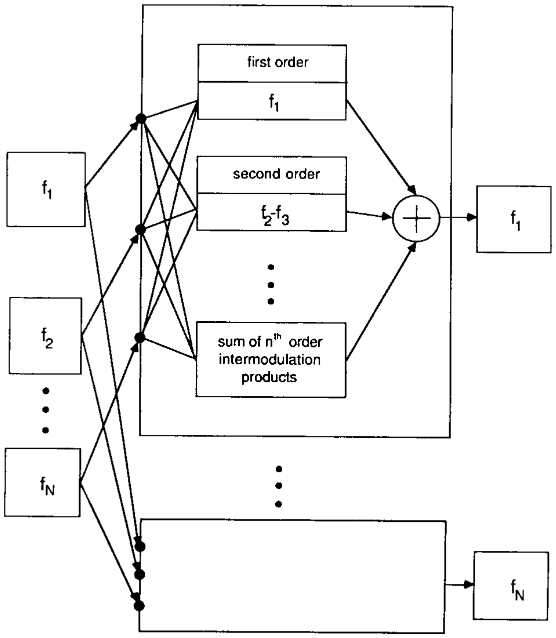
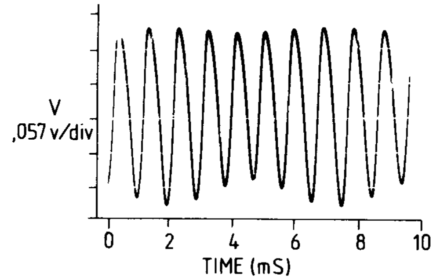


Fig. 6. Illustration of the *algebraic formula* evaluation.

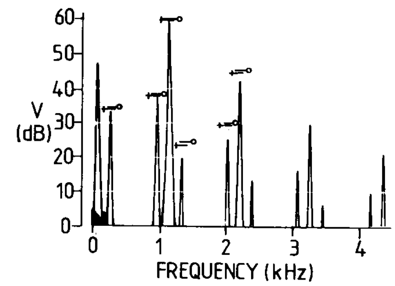
TABLE I
PARTIAL LISTING OF IPD'S WHEN dc IS NOT AN INPUT TO THE ALGEBRAIC FORMULA

Output Frequency	n	n_1	n_2	n_3
f_1, IF	1	1	0	0
	2	0	1	-1
	4	2	-1	1
	5	-1	2	-2
	7	3	-2	2
	8	-2	3	-3
f_2, LO	1	0	1	0
	2	1	0	1
	4	-1	2	-1
	5	2	-1	2
	7	-2	3	-2
	8	3	-2	3
f_3, RF	1	0	0	1
	2	-1	1	0
	4	1	-1	2
	5	-2	2	-1
	7	2	-2	3
	8	-3	3	-2
f_4, dc	0	0	0	0
	3	1	-1	1
	6	2	-2	2

As a comparison, the Sonine-expansion-based analysis was also used to simulate the diode, yielding the spectrum denoted by circles in Fig. 7(b). It is seen that the two sets of numerical results are in good agreement with each other and with the experimental results. The Sonine and power series analyses used the same set of IPD's, and in the final



(a)



(b)

Fig. 7. Voltage v_T at the diode terminals. (a) Experimental waveform. (b) Spectrum. Solid line—experimental; + —generalized power series method; ∘ —Sonine expansion method.

iteration both used 220 IPD's and up to 12th order intermodulation, with the solution calculated to a fractional accuracy of 0.1 percent. The generalized power series and Sonine expansion analyses both took 83 iterations and 5.7 s on the Digital Equipment Corporation KL10 computer to arrive at the solution using the p-factor harmonic balance scheme of Hicks and Khan [35], but only four iterations and 5.0 s using the objective function minimization method described here.

VI. CONCLUSIONS

We have presented a novel technique, termed generalized power series analysis, for analyzing nonlinear analog circuits under large-signal conditions. The technique operates entirely in the frequency domain, which enables circuits having multifrequency excitation to be simulated. The analysis progresses via minimization of an objective function and, as design specifications can be incorporated into the objective function, is ideally suited to computer-aided circuit design. The technique thus addresses many of the current problems in nonlinear microwave circuit design.

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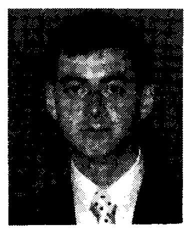

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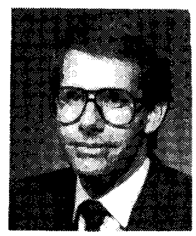

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